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A common fixed point theorem with applications to vector equilibrium problems

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ABSTRACT

In this paper, using the Brouwer fixed point theorem, we establish a common fixed point theorem for a family of set-valued mappings. As applications of this result we obtain existence theorems for the solutions of two types of vector equilibrium problems, a Ky Fan-type minimax inequality and a generalization of a known result due to Iohvidov.

1. Introduction and preliminaries

Let X be a nonempty set and Z be a convex subset of a vector space. Recall that a set-valued mapping $T : X \multimap Z$ is said to be *generalized KKM* (see [1]) if for any nonempty finite subset $\{x_1, \dots, x_n\}$ of X there is $\{z_1, \dots, z_n\} \subseteq Z$ such that $co\{z_i : i \in I\} \subseteq \bigcup_{i\in I} T(x_i)$, for each nonempty subset I of $\{1, \dots, n\}$. Inspired by this definition we introduce a new concept as follows:

Definition 1. Let X be a nonempty set, Z be a convex subset of a vector space and $\mathcal{T}$ be a family of set-valued mappings with nonempty values from X into Z. We say that $\mathcal{T}$ is *generalized equi-KKM* if for any nonempty finite subset $\{x_1, \dots, x_n\}$ of X there is $\{z_1, \dots, z_n\} \subseteq Z$ such that $\mathrm{co}\{z_i : i \in I\} \subseteq \bigcup_{i\in I} T(x_i)$, for each nonempty subset I of $\{1, \dots, n\}$ and for all $T \in \mathcal{T}$.

Remark 1. If Z is a convex subset of a topological vector space and $\mathcal{T}$ is generalized equi-KKM then, according to Lemma 3.3 in [2], for each $T \in \mathcal{T}$, $\{\overline{T(x)} : x \in X\}$ has the finite intersection property.

Example 1. Let $X = Z = [0, 1]$ and $\mathcal{T} = \{T_y : [0, 1] \multimap [0, 1]\}_{y\in[0,1]}$, where

$$T_y(x) = \begin{cases} \left[\dfrac{xy}{2}, 1\right] & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ \left[0, \dfrac{x+y}{2}\right] & \text{if } x \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

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We show that the family $\mathcal{T}$ is generalized equi-KKM. One can easily check that for all $y \in [0,1]$ we have $[\frac{1}{4}, 1] \subseteq T_y(x)$ for any $x \leq \frac{1}{2}$ and respectively, $[0, \frac{1}{4}] \subseteq T_y(x)$ if $x > \frac{1}{2}$. For any $\{x_1, \ldots, x_n\} \subseteq [0,1]$, put $z_i = \frac{1-x_i}{2}$, $1 \leq i \leq n$. We claim that

$$\mathrm{co}\{z_i : i \in I\} \subseteq \bigcup_{i\in I} T_y(x_i),$$

for each nonempty subset I of $\{1, \ldots, n\}$ and for all $y \in [0,1]$. Suppose $x_{i_1} = \min_{i\in I} x_i$ and $x_{i_2} = \max_{i\in I} x_i$. If $x_{i_2} \leq \frac{1}{2}$, then $\mathrm{co}\{z_i : i \in I\} \subseteq [\frac{1}{4}, 1] \subseteq \bigcup_{i\in I} T_y(x_i)$. If $x_{i_1} > \frac{1}{2}$, then $\mathrm{co}\{z_i : i \in I\} \subseteq [0, \frac{1}{4}] \subseteq \bigcup_{i\in I} T_y(x_i)$. If $x_{i_1} \leq \frac{1}{2} < x_{i_2}$, then $\mathrm{co}\{z_i : i \in I\} \subseteq [0,1] = T_y(x_{i_1}) \cup T_y(x_{i_2}) = \bigcup_{i\in I} T_y(x_i)$.

If X and Y are topological spaces a set-valued mapping $T : X \multimap Y$ is said to be: (i) *upper semicontinuous* (in short, u.s.c.) (respectively, *lower semicontinuous* (in short, l.s.c.)) if for every closed subset B of Y the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ (respectively, $\{x \in X : T(x) \subseteq B\}$) is closed; (ii) *closed* if its graph is a closed subset of $X \times Y$; (iii)*compact* if $\overline{T(X)}$ is a compact subset of Y.

The following lemma collects known facts about u.s.c. or l.s.c. set-valued mappings (see for instance [3] for assertion (i) and [4] for assertion (ii)).

Lemma 1. *Let X and Y be topological spaces and $T : X \multimap Y$ be a set-valued mapping.*

(i) *If Y is regular and T is u.s.c. with closed values, then T is closed.*
(ii) *T is l.s.c. if and only if for any $x \in X$, $y \in T(x)$ and any net $\{x_t\}$ converging to x, there exists a net $\{y_t\}$ converging to y, with $y_t \in T(x_t)$ for each t.*

Definition 2 (*[5]*). Let X and Y be two nonempty convex subsets of two vector spaces and V be a vector space. Let $F : X \times Y \multimap V$ and $C : X \multimap V$ be two set-valued mappings such that for each $x \in X$, $C(x)$ is a convex cone. We say that:

(i) F is $C(x)$-*quasiconvex* if for all $x \in X, y_1, y_2 \in Y$ and $y \in co\{y_1, y_2\}$, we have either $F(x, y_1) \subseteq F(x, y) + C(x)$, or $F(x, y_2) \subseteq F(x, y) + C(x)$.
(ii) F is $C(x)$-*quasiconvex-like* if for any $x \in X, y_1, y_2 \in Y$ and $y \in co\{y_1, y_2\}$, we have either $F(x, y) \subseteq F(x, y_1) - C(x)$, or $F(x, y) \subseteq F(x, y_2) - C(x)$.

It is worth mentioning that the concepts introduced above are special cases of many recent general and relaxed notions (see e.g. [6, Def. 2.5], [7, p. 1271], [8, p. 58] and [9, Def. 4.1]). By induction one can prove the following lemma (see [5] for assertion (i), respectively [10] for assertion (ii)).

Lemma 2. *Let X and Y be two nonempty convex subsets of two vector spaces and V be a vector space. Let $F : X \times Y \multimap V$ and $C : X \multimap V$ be two set-valued mappings such that for each $x \in X$, $C(x)$ is a convex cone.*

(i) *F is $C(x)$-quasiconvex if and only if for any $x \in X$, $y_i \in Y$, $1 \leq i \leq n$, $y \in co\{y_i : 1 \leq i \leq n\}$ there exists $1 \leq j \leq n$ such that $F(x, y_j) \subseteq F(x, y) + C(x)$.*
(ii) *F is $C(x)$-quasiconvex-like if and only if for any $x \in X$, $y_i \in Y$, $1 \leq i \leq n$, $y \in co\{y_i : 1 \leq i \leq n\}$ there exists $1 \leq j \leq n$ such that $F(x, y) \subseteq F(x, y_j) - C(x)$.*

2. A common fixed point theorem

Theorem 1. *Let X be a nonempty convex subset of a topological vector space, Y be a nonempty set and $T : X \times Y \multimap X$ a compact set-valued mapping satisfying the following conditions:*

(i) *for each $y \in Y$, the set $\{x \in X : x \in T(x, y)\}$ is closed;*
(ii) *the family of set-valued mappings $\{T(x, \cdot)\}_{x\in X}$ is generalized equi-KKM on Y.*

Then the family of set-valued mappings $\{T(\cdot, y)\}_{y\in Y}$ has a common fixed point, that is, there exists $x_0 \in X$ such that $x_0 \in \bigcap_{y\in Y} T(x_0, y)$.

Proof. For each $y \in Y$, put $G(y) = \{x \in X : x \notin T(x, y)\}$. Suppose that the conclusion is not true. Then $X = \bigcup_{y\in Y} G(y)$. Since the sets $G(y)$ are all open and $\overline{T(X \times Y)}$ is compact there exists a finite set $\{y_1, \ldots, y_n\} \subseteq Y$ such that $\overline{T(X \times Y)} \subseteq \bigcup_{i=1}^n G(y_i)$. Moreover, for each $x \in X \setminus \overline{T(X \times Y)}$ and $i \in \{1, \ldots n\}$, $x \in G(y_i)$, hence $X = \bigcup_{i=1}^n G(y_i)$. By (ii), there exists $\{z_1, \ldots, z_n\} \subseteq X$ such that $co\{z_i : i \in I\} \subseteq \bigcup_{i\in I} T(x, y_i)$, for each nonempty subset I of $\{1, \ldots, n\}$ and for all $x \in X$. Set $K = co\{z_1, \ldots, z_n\}$. Then $\{G(y_i) \cap K\}_{1\leq i\leq n}$ is an open cover of K. Consider a partition of unity on K, $\{\alpha_1, \ldots, \alpha_n\}$, subordinated to this open cover. Recall that this means that

$$\begin{cases} \alpha_i : K \to [0,1] \text{ is continuous}, & \text{for each } i \in \{1, \ldots, n\}; \\ \alpha_i(x) > 0 \Rightarrow x \in G(y_i) \cap K; & \\ \displaystyle\sum_{i=1}^n \alpha_i(x) = 1 \quad \text{for each } x \in K. & \end{cases}$$

Define the function $p : K \to K$ by $p(x) = \Sigma_{i=1}^{n} \alpha_i(x) z_i$. Since p is continuous function, by the Brouwer fixed point theorem there exists $x_0 \in K$ such that $x_0 = p(x_0)$. Let $I = \{i \in \{1, \ldots, n\} : \alpha_i(x_0) > 0\}$. Then $x_0 = p(x_0) \in co\{z_i : i \in I\} \subseteq \bigcup_{i\in I} T(x_0, y_i)$.

On the other hand, for each $i \in I$, since $x_0 \in G(y_i)$, $x_0 \notin T(x_0, y_i)$, hence $x_0 \notin \bigcup_{i\in I} T(x_0, y_i)$. The obtained contradiction completes the proof. □

3. Applications

Let X be a nonempty subset of a topological vector space and $f : X \times X \to \mathbb{R}$ be a function with $f(x, x) \geq 0$ for all $x \in X$. Then the scalar equilibrium problem, in the sense of Blum and Oettli [11], is to find $x_0 \in X$ such that $f(x_0, y) \geq 0$ for all $y \in X$. In the last years the scalar equilibrium problem was extensively generalized in several ways to vector equilibrium for set-valued mappings. In this paper we fix our attention on two types of vector equilibrium problems described below:

Let X be a nonempty compact convex subset of a topological vector space, Y be a nonempty set and V be a topological vector space. Let $F : X \times Y \multimap V$, $G : X \times X \multimap V$ and $C : X \multimap V$ be three set-valued mappings. Suppose that for each $x \in X$, $C(x)$ is a nonempty convex cone. Moreover, in case of problem (1), suppose that $\operatorname{int} C(x) \neq \emptyset$, for all $x \in X$. We are interested in finding a $x_0 \in X$ such that:

$$F(x_0, y) \nsubseteq -\operatorname{int} C(x_0) \quad \text{for all } y \in Y, \tag{1}$$

respectively

$$F(x_0, y) \subseteq C(x_0) \quad \text{for all } y \in Y. \tag{2}$$

These problems, or more general equilibrium problems, are studied in many papers (see, for instance, [5,7,8,12–17]) when either $X = Y$ or X and Y are distinct convex sets, each in a topological vector space. In this section existence theorems for the solutions of these problems will be obtained when Y is an arbitrary nonempty set without any algebraic or topological structure.

Theorem 2. *Suppose that the set-valued mappings F, G and C satisfy the following conditions:*

(i) *for each $x \in X$, $G(x, x) \nsubseteq -\operatorname{int} C(x)$;*
(ii) *for any $y \in Y$ there exists $z \in X$ such that $G(x, z) \subseteq F(x, y)$, for all $x \in X$;*
(iii) *G is l.s.c. on $\Delta_X = \{(x, x) : x \in X\}$ and for each $y \in Y$ the set-valued mapping $x \multimap F(x, y) - C(x)$ is closed;*
(iv) *G is $C(x)$-quasiconvex-like.*

Then there exists $x_0 \in X$ such that $F(x_0, y) \nsubseteq -\operatorname{int} C(x_0)$ for all $y \in Y$.

Proof. Define $T : X \times Y \multimap X$ by

$$T(x, y) = \{z \in X : G(x, z) \subseteq F(x, y) - C(x)\}.$$

For an arbitrary $y \in Y$ denote

$$M = \{x \in X : x \in T(x, y)\} = \{x \in X : G(x, x) \subseteq F(x, y) - C(x)\}.$$

Let $x \in \overline{M}$ and $\{x_t\}$ a net in M converging to x. Since G is l.s.c. on Δ_X, for each $v \in G(x, x)$ there exists a net $\{v_t\}$ such that $v_t \to v$ and $v_t \in G(x_t, x_t)$ for all t. Since $x_t \in M$, we have $v_t \in F(x_t, y) - C(x_t)$. Since the mapping $x \multimap F(x, y) - C(x)$ is closed, it follows that $v \in F(x, y) - C(x)$. Thus, $x \in M$, hence M is closed.

Let $\{y_1, \ldots, y_n\}$ be a finite subset of Y. By (ii), there exists $\{z_1, \ldots, z_n\} \subseteq X$ such that $G(x, z_i) \subseteq F(x, y_i)$ for each $i \in \{1, \ldots, n\}$ and all $x \in X$. Let $I \subseteq \{1, \ldots, n\}$ and $z \in co\{z_i : i \in I\}$. By (iv), for each $x \in X$ there exists $i_x \in I$ such that $G(x, z) \subseteq G(x, z_{i_x}) - C(x) \subseteq F(x, y_{i_x}) - C(x)$. Thus, $co\{z_i : i \in I\} \subseteq \bigcup_{i\in I} T(x, y_i)$, for all $x \in X$.

By Theorem 1, there exists $x_0 \in X$ such that $x_0 \in \bigcap_{y\in Y} T(x_0, y)$. If for some $y \in Y$ we would have $F(x_0, y) \subseteq -\operatorname{int} C(x_0)$ then, since $x_0 \in T(x_0, y)$, we would obtain

$$G(x_0, x_0) \subseteq F(x_0, y) - C(x_0) \subseteq -\operatorname{int} C(x_0) - C(x_0) = -\operatorname{int} C(x_0),$$

which contradicts (i). Thus, $F(x_0, y) \nsubseteq -\operatorname{int} C(x_0)$, for all $y \in Y$. □

Remark 2. If we take into account Lemma 3.2 in [18], the set-valued mapping $x \multimap F(x, y) - C(x)$ is closed whenever $F(\cdot, y)$ is u.s.c. with nonempty compact values and C is closed.

Theorem 3. *Suppose that the set-valued mappings F, G and C satisfy the following conditions:*

(i) *for each $x \in X$, $G(x, x) \subseteq C(x)$;*
(ii) *for any $y \in Y$ there exists $z \in X$ such that $F(x, y) \subseteq G(x, z)$, for all $x \in X$;*
(iii) *the set-valued mapping $x \multimap G(x, x) + C(x)$ is closed and for each $y \in Y$, $F(\cdot, y)$ is l.s.c.;*
(iv) *G is $C(x)$-quasiconvex.*

Then there exists $x_0 \in X$ such that $F(x_0, y) \subseteq C(x_0)$ for all $y \in Y$.

Proof. Define $T : X \times Y \multimap X$ by

$$T(x, y) = \{z \in X : F(x, y) \subseteq G(x, z) + C(x)\}.$$

We prove that T satisfies the requirements of Theorem 1. For an arbitrary $y \in Y$ denote

$$M = \{x \in X : x \in T(x, y)\} = \{x \in X : F(x, y) \subseteq G(x, x) + C(x)\}.$$

Let $x \in \overline{M}$ and $\{x_t\}$ a net in M converging to x. Since $F(\cdot, y)$ is l.s.c., for any $v \in F(x, y)$ there exists a net $\{v_t\}$ such that $v_t \to v$ and $v_t \in F(x_t, y)$ for all t. Since $x_t \in M$, we have $v_t \in G(x_t, x_t) + C(x_t)$. Since the mapping $x \multimap G(x, x) + C(x)$ is closed, it follows that $v \in G(x, x) + C(x)$. Thus, $x \in M$, hence M is closed.

Let $\{y_1, \dots, y_n\}$ be a finite subset of Y. By (ii), there exists $\{z_1, \dots, z_n\} \subseteq X$ such that $F(x, y_i) \subseteq G(x, z_i)$ for each $i \in \{1, \dots, n\}$ and all $x \in X$. Let $I \subseteq \{1, \dots, n\}$ and $z \in co\{z_i : i \in I\}$. By (iv), for each $x \in X$ there exists $i_x \in I$ such that $G(x, z_{i_x}) \subseteq G(x, z) + C(x)$. Hence, $F(x, y_{i_x}) \subseteq G(x, z_{i_x}) \subseteq G(x, z) + C(x)$. This implies $z \in T(x, y_{i_x}) \subseteq \bigcup_{i\in I} T(x, y_i)$, hence $co\{z_i : i \in I\} \subseteq \bigcup_{i\in I} T(x, y_i)$.

By Theorem 1, there exists $x_0 \in X$ such that $x_0 \in \bigcap_{y\in Y} T(x_0, y)$. Then, for each $y \in Y$ we have

$$F(x_0, y) \subseteq G(x_0, x_0) + C(x_0) \subseteq C(x_0) + C(x_0) = C(x_0). \quad \square$$

Theorems 2 and 3 are different from other close results from [5,7,12,14,15,17] by conditions (ii) and (iii), proof techniques and, especially, by the fact that Y is a set without any algebraic or topological structure. The applicability of Theorem 3 is put into evidence by the following example.

Example 2. Let Y be a nonempty subset of the interval $[1, \infty)$, $X = [0, 1]$, $V = \mathbb{R}$,

$$G(x, z) = \begin{cases} (-\infty, z - x] & \text{if } z \le \frac{1}{2}, \\ (-\infty, x - z] & \text{if } z > \frac{1}{2}, \end{cases}$$

$$F(x, y) = (-\infty, 1 - xy] \quad \text{and} \quad C(x) = (-\infty, 0].$$

Condition (ii) in Theorem 3 is fulfilled since, for each $y \in Y$, $F(x, y) \subseteq G(x, 1)$, for all $x \in [0, 1]$. One can readily verify that all the other requirements of the same theorem are satisfied. By direct checking one can see that $x_0 = 1$ satisfy the conclusion of Theorem 3. Since the set Y is not necessarily convex any other known result is not applicable.

Theorem 4. *Let X be a nonempty compact convex subset of a topological vector space and Y be a nonempty set. Let $f : X\times Y \to \mathbb{R}$, $g : X \times X \to \mathbb{R}$ be two functions and $a \in \mathbb{R}$. Suppose that:*

(i) *$g(x, x) \le a$ for all $x \in X$;*
(ii) *for each $y \in Y$ there is $z \in X$ such that $f(x, y) \le g(x, z)$ for all $x \in X$;*
(iii) *f is l.s.c. in the first variable and g is u.s.c. on Δ_X;*
(iv) *g is quasiconcave in the second variable.*

Then there exists $x_0 \in X$ such that $f(x_0, y) \le a$ for all $y \in Y$.

Proof. Take in the previous theorem $V = \mathbb{R}$,

$$F(x, y) = (-\infty, f(x, y) - a], \qquad G(x, z) = (-\infty, g(x, z) - a], \qquad C(x) = (-\infty, 0].$$

It can be readily shown that if $h : X \to \mathbf{R}$ is a u.s.c. (respectively, l.s.c.) function, then the set-valued mapping $H : X \multimap \mathbf{R}$, defined by $H(x) = (-\infty, h(x)]$, is u.s.c. (respectively, l.s.c.). Consequently, under condition (iii), $G|_{\Delta_X}$ is u.s.c. and for each $y \in Y$, $F(\cdot, y)$ is l.s.c.. Moreover, since a t.v.s. is regular, by Lemma 1(i), the set-valued mapping $x \multimap G(x, x) = G(x, x) + C(x)$ is closed. Thus, condition (iii) in Theorem 3 holds. We check condition (iv) from the same theorem. Let $x \in X$, $z_1, z_2 \in X$ and $z \in co\{z_1, z_2\}$. Since g is quasiconcave in the second variable, for each $x \in X$, $g(x, z) - a \ge \min\{g(x, z_1) - a, g(x, z_2) - a\}$. Thus, for each $x \in X$ there is an index $i \in \{1, 2\}$ such that $G(x, z_i) \subset G(x, z) = G(x, z) + C(x)$.

It is easy to verify that all the other conditions of Theorem 3 are satisfied and the desired conclusion follows from this theorem. $\square$

By Theorem 4 we derive the following Ky Fan-type minimax inequality:

Theorem 5. *If conditions* (ii), (iii) *and* (iv) *in Theorem 4 hold, then*

$$\inf_{x\in X} \sup_{y\in Y} f(x, y) \le \max_{x\in X} g(x, x).$$

Proof. Take in the previous theorem $a = \max_{x\in X} g(x, x)$. $\square$

Definition 3. Let X be a convex set in a vector space and V be a vector space. A function $f : X \to V$ is said to be *almost affine* if for each $x_1, x_2 \in X$ and $\mu \in [0, 1]$ there exists $\lambda \in [0, 1]$ such that $f(\mu x_1 + (1 - \mu)x_2) = \lambda f(x_1) + (1 - \lambda)f(x_2)$.

Remark 3. (a) If I is a real interval any monotone function $f : I \to \mathbf{R}$ is almost affine.

(b) If $f : X \to V$ is almost affine, then one can easily prove that for any $x_1, \ldots, x_n \in X$ and $x \in co\{x_1, \ldots, x_n\}$ there exist $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $f(x) = \sum_{i=1}^n \lambda_i f(x_i)$.

As an application of Theorem 1, we give a slight generalization of a known result, due to Iohvidov [19].

Theorem 6. *Let X be a nonempty compact convex subset of locally convex Hausdorff topological vector space E and $g : X \times X \to E$ be a continuous function. Suppose that:*

(i) *for each $x \in X$, $g(x, \cdot)$ is almost affine;*
(ii) *for each $x \in X$ there exists $y \in X$ such that $g(x, y) = 0$.*

Then there exists $x_0 \in X$ such that $g(x_0, x_0) = 0$.

Proof. Let $\mathscr{P}$ be a sufficient family of continuous seminorms on E generating the topology of E. Denote by

$$F_p = \{x \in X : p(g(x, x)) = 0\}, \quad p \in \mathscr{P}.$$

Since the family $\mathscr{P}$ is sufficient, a point $x_0 \in X$ satisfies the conclusion of the theorem if $x_0 \in \bigcap_{p \in \mathscr{P}} F_p$. Since X is compact and F_p are closed sets, it suffices to prove that for any nonempty finite subset $\{p_1, \ldots, p_k\}$ of $\mathscr{P}$, $\bigcap_{j=1}^k F_{p_j} \neq \emptyset$. Define the set-valued mapping $T : X \times X \multimap X$ by

$$T(x, y) = \left\{ z \in X : \sum_{j=1}^k p_j(g(x, z)) \leq \sum_{j=1}^k p_j g(g(x, y)) \right\}.$$

It is clear that for each $y \in X$, the set $\{x \in X : x \in T(x, y)\}$ is closed. We show that for each $x \in X$, $T(x, \cdot)$ is a *KKM* mapping. Let $\{y_1, \ldots, y_n\} \subseteq X$, $I \subseteq \{1, \ldots, n\}$ and $y \in \mathrm{co}\{y_i : i \in I\}$. Then, by Remark 3(b), there exists $\lambda_i \geq 0$ (depending on x), with $\sum_{i \in I} \lambda_i = 1$, such that $g(x, y) = \sum_{i \in I} \lambda_i g(x, y_i)$. We have

$$\begin{aligned} \sum_{j=1}^k p_j(g(x, y)) &= \sum_{j=1}^k p_j \left(\sum_{i \in I} \lambda_i g(x, y_i) \right) \leq \sum_{j=1}^k \sum_{i \in I} \lambda_i p_j \left(g(x, y_i)\right) \\ &= \sum_{i \in I} \lambda_i \left(\sum_{j=1}^k p_j \left(g(x, y_i)\right) \right) \leq \max_{i \in I} \sum_{j=1}^k p_j \left(g(x, y_i)\right). \end{aligned}$$

Hence $co\{y_i : i \in I\} \subseteq \bigcup_{i \in I} T(x, y_i)$. By Theorem 1 there exists $x_0 \in X$ such that $x_0 \in T(x_0, y)$, for all $y \in X$. By (ii), there is $y_0 \in X$ such that $g(x_0, y_0) = 0$. Since $x_0 \in T(x_0, y_0)$, $\sum_{j=1}^k p_j \left(g(x_0, x_0)\right) \leq 0$, $x_0 \in \bigcap_{j=1}^k F_{p_j}$. $\square$

Remark 4. If $f : X \to X$ is a continuous function, taking $g(x, y) = f(x) - y$, hence Theorem 6 reduces to the well-known Tychonoff fixed point theorem.

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